OUTSTANDING CREATIVITY AWARD FOR THE NATURAL SCIENCES 2016/17

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First of all, I would like to express my gratitude towards the CUHK Convocation for granting me the Outstanding Creativity Award for the Natural Sciences 2016/17. I am glad that my effort and creativity are appreciated and recognized.

I spent the summer right after my third year conducting original mathematical research at Williams College. I researched on four different number theoretic problems. This experience was challenging yet rewarding. I am delighted to be a member of such an amazing research group. I would also like to thank my research advisors Prof. Steven J. Miller, Prof. Eyvindur Ari Palsson and Dr. Tian An Wong for their support and advice. I submitted two papers for publications ([1], [2]). The papers of the other two projects are in preparation. In this article, I will focus on my completed work.

Number theory is described by many as the *'Queen of Mathematics'* because of its beauty in the purest form and its fundamental importance in mathematics. It is devoted to study the basic properties of numbers. It is often said 'the simpler the more challenging it is'. Number theory is an excellent example of this saying. There are many simple notions that we may have come across in even elementary schools, such as prime numbers. However, one has to admit that we are far from understanding them fully. There is still a large number of open problems in which mathematicians have few clues on how to approach them. However, number theory remains an active research area in mathematics for millennia, mostly driven by curiosity and aim for better understanding of the Nature. It is also a constant source of excitement and fascination for many prominent mathematicians.

My first project is an analytic look at the *functoriality conjecture*. This was proposed by Langlands and it asserts that given a homomorphism between the L-groups of two reductive groups, there corresponds a transfer of automorphic forms. This conjecture has profound consequences and it is a grand unification of many distinct branches of mathematics. A recent strategy is known as *Beyond Endoscopy* (2004). The main idea is to detect the functorial transfer by the pole of automorphic L-functions at $s = 1$ and the trace formula. Altuğ completed the analysis of standard representation on $GL(2)$ in 2012 by using techniques from analytic number theory.

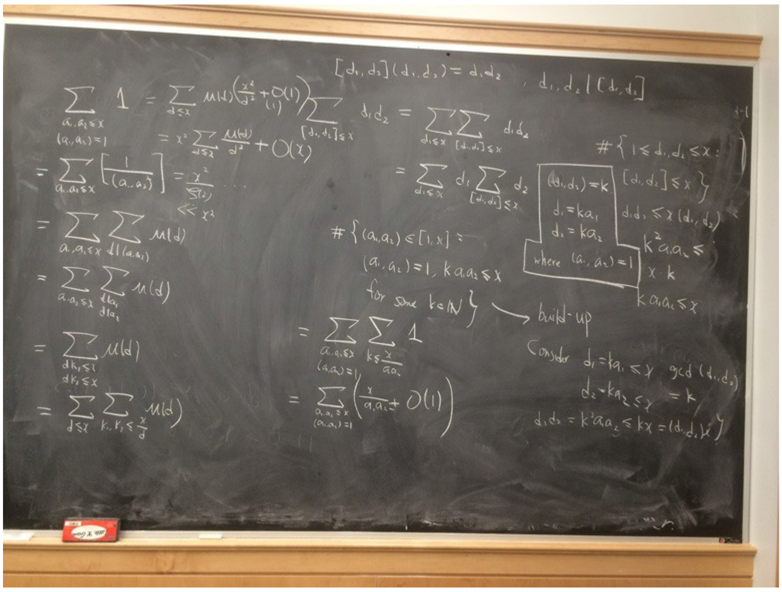
We attempted to carry out Altuğ's analysis to general $GL(n)$. We were able to recast Arthur-Selberg trace formula and apply approximate functional equation of Artin's L-functions to smooth out singularities of real orbital integral. One of the major differences from Altuğ's case is that we have to work with a different L-function involving Galois representations. This poses significant difficulties in carrying out further analysis as some parts of Altuğ's work requires fine care, such as the application of Poisson summation formula. We would like to continue further investigations along this line.

My second project is about generalizations of perfect numbers. A natural number is said to be *perfect* if it is the sum of all of its *proper* divisors. For instance, 6 is perfect (all proper divisors are 1, 2, 3 and $6 = 1 + 2 + 3$). In fact, mathematicians in ancient Greeks were already thinking seriously about them, such as Euclid found a way to generate *even* perfect numbers. It is natural to ask whether the list of perfect numbers go on and whether there is any odd perfect number. These two questions can be easily understood, but their truth remains one of the biggest mystery in number theory.

However, developments in perfect numbers have not slowed down. Mathematicians started to look at *generalizations* of perfect numbers. Pollack and Shevelev introduced the notion of near-perfectness in 2012. Given a natural number k, a natural number is said to be *k-near-perfect* if it is a sum of all of its proper divisors with *at most k exceptions*. They established an upper bound of the number of k-near-perfect numbers in $[1, x]$, which is $\frac{x}{\log x}(\log\log x)^{k-1}$.

We showed that the number of k-near-perfect numbers is *of exact order* $\frac{x}{\log x}(\log\log x)^{[\frac{\log(k+4)}{\log 2}]-3}$ for $k \geq 4$. This substantially improves the known results. Moreover, we extended the notion by allowing the number of exceptional divisors growing with n. We showed if k is a function less than $(\log x)^{\epsilon}$ for some $\epsilon \in (0, \log 2)$, then the set of k-near-perfect numbers has density 0; if k is a function larger than $(\log x)^r$, where $r > \log 2$, then the set of k-near-perfect numbers has positive lower density.

Photo 1:



This photo was taken during an evening math discussion/brainstorm section.

Photo 2:

This photo was taken during the progress report section with my amazing groupmates.
(I am the middle one.)

References

[1] O. Gonzalez, C. H. Kwan, S.J. Miller, R. van Peski and T.A. Wong, *On smoothing singularities of elliptic orbital integrals on GL(n) and Beyond Endoscopy*, submitted to the *Journal of Number Theory*, Arxiv: https://arxiv.org/abs/1608.05938.

[2] P. Cohen, K. Cordwell, A. Epstein, C. H. Kwan and A.Lott, S.J. Miller, *On within-perfectness and near-perfectness*, submitted to *Acta Arithmetica*, Arxiv: https://arxiv.org/abs/1610.04253.